TOWARD A COMMOGNITIVE COMPLEX SYSTEMS PERSPECTIVE: NETWORKING KNOWLEDGE IN PIECES AND COMMOGNITION

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Abstract

Knowledge in Pieces (KiP) and Commognition are two influential heuristic epistemological frameworks. While the two approaches have different historical origins and distinct theoretical foci (encoding of intuitive knowledge and a complex systems approach to understanding conceptual development in the case of KiP and mathematical discourse and its development in individual and communal forms in the case of Commognition) they both share an accountability to the nuances of the moment-by-moment details of records of cognition. Both perspectives draw insights from fine-grained analysis of reasoning from video data of learning interactions and reasoning processes. In terms of the kinds of theoretical machinery produced, both KiP and Commognition share a skepticism towards common sense terms about knowledge and they both emphasize the need for precision in new theoretical constructs and models. This theoretical paper explores both the differences and similarities between Commognition and Knowledge in Pieces and begins to map out the possibilities and potential benefits for networking these perspectives.

Keywords: Mathematics Education, Networking theories, Knowledge in Pieces, Commognition

INTRODUCTION

Mathematics education draws upon a wide range of theoretical perspectives with diverse epistemological and methodological commitments. This diversity in theoretical perspectives raises questions for researchers about how to best leverage this range in perspectives (Artigue et al., 2006; Radford, 2008). To engage with the range of perspectives, researchers have been exploring the approach of networking theoretical perspectives (Bikner-Ahsbahs & Prediger, 2014; Scheiner & Bosch, 2025). The Networking of theories approach acknowledges the complexity and multifaceted nature of phenomena such as mathematical knowledge and learning processes and encourages using multiple theoretical lenses to capture this complexity while at the same time respecting the individual identities and specificities of different theoretical perspectives.

Thus far, this work has been more frequently carried out between perspectives with similar underlying epistemological assumptions but different research foci, as opposed to by examining potential points of contact and divergence between "opposing" perspectives (Scheiner, 2022). While complementary analyses of data from perspectives with convergent theoretical commitments are productive, Scheiner (2020) claims

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that far less attention has been paid to opportunities for theory building and theory advancement that are offered by considering the tensions between theoretical perspectives with potentially conflicting or fundamentally different theoretical commitments.

This piece aims to promote dialogue between two communities that on the surface appear to be guided by opposing theoretical perspectives: Knowledge in Pieces (diSessa, 1993) and Commognition (Sfard, 2008). While differing in their epistemological assumptions, both have led to theoretical contributions related to students' mathematical thinking and learning through fine-grained analyses of the moment-by-moment details of learning.

In the next sections, we first describe the central tenets and commitments of both perspectives. We then turn to a description of work that is guided by Knowledge in Pieces and uses the model of knowledge as a complex system to study a social and discursive phenomenon: the formation of personal ideologies. This example provides the bridge to a proposal for a way Knowledge in Pieces and Commognition could be productively used together to analyze the moment-by-moment details of learning that are situated in culture and history. We then articulate a vision for a Commognitive Complex Systems Perspective and briefly elaborate how this joint perspective offers theoretical machinery to make sense of the dynamics of in the moment reasoning and processes of change over time, and the embodied source of mathematical knowledge, a strength of Knowledge in Pieces, while also recognizing the cultural and historical source and genesis of elements of knowledge, a strength of the Commognitive approach. We close with a discussion of what we see as productive next steps for developing this joint perspective theoretically and methodologically, as well as what essential tensions remain between the perspectives.

COMMOGNITION

Sfard's influential monograph on commognition (Sfard, 2008) introduced the novel perspective that mathematics can be conceptualized as a historically established discourse, and learning mathematics involves becoming a participant in this discourse. Sfard's work is informed by the work of learning theorists and philosophers who have proposed that learning involves becoming a participant in historically established forms of activity and that thought and language are inseparable (Vygotsky, 1987; Wittgenstein, 1953).

As an example of a cultural-historical perspective, commognition is strongly shaped by the issue of how knowledge is developed at a societal level and how individuals learn this culturally accumulated knowledge. A commognitive analysis of learning involves studying the historical development of a mathematical concept and where there are critical historical transitions in the understanding of that concept. Commognitive researchers do not claim that students will necessarily have to pass through the same trajectory as were passed through in the historical development of concepts, though to commognitive researchers these historical trajectories give clues about where existing discourses needed to be incorporated into or transcended by new discourses, together with their new keywords, visual mediators, routines, and eventually narratives. These parallels between historical shifts in discourse and shifts in student learning have been elaborated in studies such as Caspi & Sfard (2012) studying students' transition to algebraic discourse and Lavie & Sfard (2019) investigating young students' transition to a quantitative numerical discourse.

Mathematics is a particularly interesting case to conceptualize through this perspective because Sfard explains how mathematics as a discipline is a special discursive system because it is autopoietic. That is, it creates the objects participants talk about. Mathematics discourse is identified through its characteristic keywords, visual mediators (such as numerals, symbols, and graphs), narratives (such as axioms, theorems, and conventions) and routines (well-defined repetitive patterns of mathematical actions, such as adding fractions, are performed). Some authors adopting the commognitive perspective include communicational acts such as gestures within the category of visual mediators, whereas others believe that it would be productive to recognize nonverbal communication acts as another category (Sinclair, 2022).

From a commognitive standpoint, learning mathematics is a process of individualizing communal mathematical discourse and coming to be able to participate in the discursive processes of a mathematics community. This involves creating narratives one can endorse (such as theorems and definitions, even if they are not new on the communal level). Commognition assumes a social source of knowledge: that is, the key process of individualization within commognition is, for the case of routines, being able to enact individually an activity that first was only able to be performed with others. One's own personal mathematical discourse changes through participation in a discourse community. On the other hand, through participation in collective discourse practices, an individual can also effect change on the discourse practices of a community. The commognitive theoretical perspective offers ways to think about an issue traditionally raised by conceptual change researchers: Why are some mathematical ideas persistently difficult to teach and learn? To account for this, the commognitive perspective posits that the compression of layers of process oriented and object level discourse over historical time is at root. Sfard documented a recurring progression in the development of mathematics, both as a discipline and in the ways individuals come to understand it, from referring to mathematical ideas as processes and then later being able to "reify" these processes so that one can also talk about and work with the ideas as if they are objects. Eventually, both ways of engaging with mathematical concepts are available to learners, depending on context.

In carrying out this process of individualization of cultural-historical constructs, commognition posits two main kinds of learning processes: object-level learning and meta-level learning. Object-level learning is about students coming to endorse an increasing number of narratives, the objects of which they are already familiar with. These narratives involve routines and meta-rules they are familiar with. Object-level discourse is considered easy to learn and cumulative. However, not all learning involves familiar objects, routines, and meta-rules. When new objects (e.g., negative numbers) or new meta-rules (need to justify narratives based on theorems or axioms) are introduced (Sfard, 2007), a new kind of learning is required: transitioning to a new discourse with new rules (Sfard, 2007). Meta-level learning involves coming in contact with an incommensurable discourse (i.e. the new discourse involves different criteria for deciding which narratives should be endorsed than the initial discourse). From a commognitive standpoint, such a transition cannot be internally resolved and requires a person more adept in the discourse to guide the process.

Since the publication of Sfard's book on Commognition, there have been influential special issues on the application and development of the commognitive perspective within leading journals (e.g., Cooper and Kontorovich, 2022; Tabach & Nachieli, 2016). Thus far, the commognitive perspective has been used to study learning of a range of mathematical topics (complex numbers, square roots, arithmetic series, plane geometry, tangents, matrices, algebra, fractions, multiplying integers, functions) and transcended age of participants (small children, students at elementary, middle and high school, as well as undergraduate mathematics students and pre- and in-service teachers).

KNOWLEDGE IN PIECES

diSessa's (1993) monograph "Toward an Epistemology of Physics" introduced a novel way to think about the nature and form of intuitive knowledge and its role in thinking and learning processes. Beyond demonstrating the productive role that intuitive knowledge plays in the reasoning processes of both novices and experts, a major contribution of diSessa's line of theorizing has been advancing the view that knowledge can be productively modeled as a complex system composed of many different kinds of elements. These elements of the system are contextually activated, meaning they are cued variously depending on the sense-making demands of a given context and the prior history of their activation by a learner. They can be productive in some contexts, and they are the foundation for expert knowledge. Novice knowledge tends to be cued less reliably, being activated in contexts where it is sometimes productive, and other times not. Expert knowledge, on the other hand, tends to be activated in contexts where it is productive, and in productive coordination with other elements. The transition from novice to expert (i.e., learning) is viewed as a gradual process of "tuning toward expertise," through which the knowledge system is reorganized and refined. Conceptualizing knowledge and learning in this way has important implications for instruction. In particular, in contrast to approaches that assume that change occurs through a process of wholesale replacement of a misconception, instruction from a Knowledge in Pieces perspective seeks to engage students' fine-grained knowledge resources and help students recognize their use more appropriately.

Initial work developing the Knowledge in Pieces perspective focused primarily on abstractions of experience interacting in the physical world, which led to the articulation of the construct of phenomenological primitive (p-prim) (diSessa, 1993). P-prims are sub-conceptual knowledge elements, the function of which is to account for individuals' expectations about how the physical world works. P-prims are evoked as a whole and are "self-explanatory," accounting for comfort in one situation and surprise in another. A large corpus of p-prims was empirically identified, including discussion of their source, developmental history, encoding, and local processes around their invocation and use. A prototypical example of a p-prim is Ohm's prim, which schematizes the experience that exerting more effort results in more effect (and more resistance leads to less effect).

Since the publication of diSessa's monograph, Knowledge in Pieces has been taken up and developed in the study of mathematics thinking and learning, making a variety of contributions. Contributions of Knowledge in Pieces to research in learning mathematics involve: algebraic thinking (Izsák, 2000; Levin, 2018; Levin & Walkoe, 2022), probability (Abu-Ghalyoun, 2021; Wagner, 2010), integration (Jones, 2013), limits (Adiredja, 2021), exponential functions (Allahyari, 2024), rectangular area (Izsák, 2005) and proportional reasoning (Izsák, Beckman, & Stark, 2022). Additionally, a number of Knowledge in Pieces-grounded studies with prospective mathematics teachers have been conducted in recent years (e.g., Francom, 2022; Leitch, 2023; Witt, 2023).

When describing Knowledge in Pieces, expositions often focus on the constitutive elements (p-prims) and focus on Knowledge in Pieces as a theoretical perspective to help us understand intuitive knowledge. However, this characterization highlights only a thin slice of the perspective and does not highlight the focus on uncovering new forms of knowledge at both the element and ensemble level. In terms of ensembles of elements, the most well-articulated kind of knowledge system described by Knowledge in Pieces, coordination classes, was created to help explain the kind of work that a concept like that of force needs to do (diSessa & Sherin, 1998). The basic phenomenology that a coordination class aims to model is that understanding a concept (canonically a physical quantity like force) means that one needs to be able to appropriately recognize, get information about and operate with the concept across a wide variety of contexts. A well-functioning knowledge system like a coordination class must have two important features: span (the ability to cover a wide number of contexts or representations of the concept) and alignment (the ability to come to the same determination of the quantity regardless of the information able to be extracted in a context). Structurally, coordination classes have two distinct parts, a perceptual part and an inferential part. The perceptual part accounts for how individuals get information about a concept (e.g., What does one attend to in order to read out relevant information about functions when they are presented in graphical, numerical, or symbolic forms?) and the inferential part involves all the inferences tied to making the determination (e.g., How does one actually go about using the information one reads out of a context to figure out the value of a function at a point?) P-prims, as units of explanation about a phenomenon, are part of the inferential network. In this way, the discussion of elements such as p-prims and knowledge systems such as coordination classes are linked. Levin (2018) adapted and extended the coordination class model to describe the interplay between perception, inference, and action involved in mathematical problem solving.

Having discussed the core "reference models" of Knowledge in Pieces (diSessa & Levin, 2021; Swanson & Levin, 2022), we note that diSessa has long called for other forms of knowledge to be articulated and considered as part of knowledge systems (diSessa, 1996). In particular, the extent to which mathematical knowledge is embodied is a question that is being actively investigated by psychologists and neuroscientists, in an effort to better understand the essence of human cognition (Dehaene, 1997; Varela, Thompson & Rosch, 2017). Additionally, we might consider that some kinds of mathematical reasoning, especially in contexts that involve making mathematical models of how the world works, might draw upon p-prims. In the landmark text introducing an embodied perspective on mathematical thinking, *Where Mathematics Comes From*, Lakoff and Nuñez (2000) describe how many abstract mathematical topics such as arithmetic operations, algebra, logical relations, infinity, and conceptualizations of space can be conceptualized via metaphors. These metaphors and related constructs such as image schemata could be a corpus of intuitive elements that comprise mathematical knowledge systems. Chiu (2000) elaborates some of these potential connections between mathematical intuitions and conceptual metaphors.

However, a broader contribution of Knowledge in Pieces beyond recognizing the role that intuitive knowledge can play in reasoning processes is the very idea of modeling thinking and learning using a complex systems perspective. In this line of thinking the elements of the system are not required to have an identifiably heterogeneous ontology. The elements can still be thought of as neither inherently right nor wrong, but instead activated productively or unproductively according to context. For example, Levin & Walkoe (2022) describe "seeds" of algebraic thinking which preserves the focus on knowledge systems with many different kinds of elements, not necessarily mathematical p-prims in particular. As a middle ground between completely not focusing on ontology and focusing on a specific type of knowledge ontology, Kapon and diSessa generalized the "self-explanatory" function of phenomenological primitives to elements of students' knowledge system she calls "explanatory primitives" or e-prims (Kapon & diSessa, 2012). Like phenomenological primitives, explanatory primitives are elements of sense-making that explain students' comfort or surprise as they reason. However, explanatory primitives do not necessarily arise from physical experience. The source of an explanatory primitive can be one's instructional experience in classes. Kapon et al. (2015), for instance, uses the notion of e-prims to describe learning processes involved in learning probability.

This notion of an explanatory primitive is of particular interest in the current paper where part of our proposal for networking of perspectives will involve illustrating how one could use the complex systems approach of Knowledge in Pieces with mathematical objects described by Commognition.

SIMILARITIES AND DIFFERENCES BETWEEN PERSPECTIVES

Having introduced both of the main epistemological perspectives, we summarize some of the main points of comparison and contrast in Table 1 in order to set the stage for our analysis of potentials for dialogue between them. The dimensions of comparison we have chosen illustrate the broadly shared focus of theorizing, the similarities in kinds of data collected and analyzed, and even the shared commitment to developing a new technical vocabulary from grounded analyses of moment-by-moment process data. The dimensions of contrast illustrate the differing assumptions the perspectives have regarding the nature of knowledge, its source and organization, as well as the approach each perspective currently takes for capturing knowledge-in-transition.

Table 1. Comparison Dimensions of Commognition and KiP

Comparative categories	Commognition	KiP
COMMONALITIES		
Focus of theorizing	Nature and form of (mathematical) knowledge and its development in individual and communal forms	Nature and form of individuals' intuitive knowledge, based on experiences, the role of such knowledge in reasoning and learning.
Methodological approach	Analysis of moment-by-moment reasoning processes (e.g., interviews, joint work on tasks, classroom discussions) coordinated with historical analysis	Analysis of moment-by-moment reasoning processes (e.g., interviews, joint work on tasks, classroom discussions)
Critiques leveled at other perspectives	Use of terms with multiple colloquial meaning in theorizing, thereby creating the need for the development of a new technical vocabulary	
DIFFERENCES		
Source of knowledge	Social and cultural practices that are then individualized	Heterogeneous, but canonically individuals constructing knowledge structures based on experiences, especially of the physical world

Nature of elements	Discursive elements, such as mathematical objects, narratives and routines	Heterogeneous, including embodied schemes, narratives
Nature of knowledge organization	Multi-layered	Complex systems perspective
Attention to change process	Phase models involving possibly abrupt shifts between incommensurable discourses following commognitive conflict	Continuous tuning based on feedback of activation priorities of elements in a complex knowledge system
The role of context	Construal of context as important more recently documented	Knowledge activation is context dependent, consistently documented across studies

AN EXAMPLE OF STUDYING DISCOURSE THROUGH A KIP LENS

In this section, we begin moving towards bringing commognition and Knowledge in Pieces into contact by providing an example of how Knowledge in Pieces as a heuristic epistemological framework can influence a domain far from its original focus on intuitive knowledge about the physical world, in particular one that is social and discursive in its origins.

While intuitive knowledge and p-prims in particular have been a canonical focus of research within the KiP framework, an interest in the nature of intuitive knowledge more generally extends to domains far beyond individuals' sense of physical mechanism. For example, inspired by the self-explanatory function of p-prims, Philip (2011) created the construct of naturalized axioms (Philip, 2011) describing the intuitive knowledge that comprises individuals' ideologies. The term "axiom" connotes that these elements, as with p-prims, are taken to be true and self-evident in particular contexts. "Naturalized" connotes that they are socially constructed. Naturalized axioms gain the warrant of common sense within particular historical, cultural, and social contexts and are used as if they were natural, inevitable, universal, and ahistorical. Examples of naturalized axioms that are used when making sense of schooling include: "The harder you try, the more likely you are to succeed," "Some kids are just smart," "Inequality will always exist," "Competition is good." Philip gives the examples of others including "If a teacher doesn't care for her students, it's more difficult for them to learn" and "It's difficult for students to learn if their basic needs aren't met." Thus, in the case of naturalized axioms, p-prims were inspirational in thinking about intuitions and knowledge systems that have their source in social and cultural experiences, as opposed to physical ones.

Not only was the focus on elements and systems generally important for Philip, but he also imported the focus of KiP on knowledge system dynamics (in particular, the construct of a coordination class; diSessa & Sherin, 1998) to help him understand ideology change processes. In particular, Philip described a case study of Alan, a White beginning teacher, that illustrated how ideological change can be explained as a dynamic and uneven process within a *system of knowledge*. Through discourse analysis of Alan's participation in a teacher professional learning group focused on social justice, Philip traced how Alan's ideological reasoning shifted across moments—sometimes reinforcing deficit-oriented or colorblind assumptions, and at other times raising justice-oriented commitments. Rather than interpreting these fluctuations as inconsistency on the part of Alan, Philip framed them as evidence of the *piecemeal* nature of ideology, where ideological knowledge elements were selectively activated or suppressed depending on context. This context-sensitive activation resonates with the KiP notion of system tuning and coordination class structure, emphasizing that ideological development involves not merely the replacement of old beliefs with new ones, but the reorganization of how intuitive and discursive elements work together across social activity. Furthermore, Philip's portrayal of Alan's concept of "teachers blaming students" exemplified a complex system that his naturalized axioms were elements of.

The social source and discursive nature of self-explanatory elements like naturalized axioms bears similarity to Sfard's notion of narratives that individuals and groups endorse through their activity. Thus, it seems reasonable that as researchers we be inspired by Philip's approach of using p-prims and coordination classes as reference models to theorize about complex systems where the elements of the system are discursive elements that gain their meaning through interactions with others and where the system is tuned through the feedback individuals get through interaction with others. In the next section, we turn to a discussion in the context of analyses of mathematical reasoning processes of what can be gained through bringing KiP and Commognition in contact with each other.

DIALOGUING BETWEEN KIP AND COMMOGNITION IN THE CONTEXT OF MATHEMATICAL LEARNING PROCESSES

In this section, we ground our discussion of the dialogue between Commognition and KiP approaches by presenting what we see as issues KiP would highlight in two recently published commognitive analyses of mathematical learning (Ben-Dor & Heyd-Metzuyanim (2021); Cooper & Lavie, 2021).

Example 1: Students in the process of shifting ways of reasoning through task engagement

Cooper and Lavie (2021) frame moments of learning as bridging between incommensurable discourses and argue that carefully crafted task situations can support this bridging through interdiscursivity. We discuss here data from one of the three contexts they discuss in their paper regarding non-positive exponents (e.g., reasoning about 2^0), where students are asked to apply exponent rules at the edge of their scope, surfacing criteria for endorsing narratives and enabling a shift from ritualized participation ("doing what counts") to explorative participation ("why it counts").

A typical prompt has students treat 2^0 as an unknown, x, and apply exponent laws: $2^3 \cdot 2^0 = 2^3$, which can be rewritten as $8 \cdot x = 8$, allowing one to solve for $x = 1$. While this yields the conventional answer, it simultaneously presumes the very extension under debate, encouraging reflection on the legitimacy and scope of mathematical narratives. Commognitively, this is a meta-level shift, as students interrogate the boundaries of acceptable reasoning. From a KiP lens, the intuitive idea of "multiplication makes things bigger" may be a knowledge element that is activated in this context but in need of shifting prioritization. In this case, the idea "exponent means repeated multiplication" along with "multiplication makes things bigger" are reorganized and reweighted in relation to new forms of discourse-based reasoning. Thus, the exponent case exemplifies how discursive change and the tuning of knowledge elements can be analyzed together, showing the productive potential of networking these perspectives.

The Cooper and Lavie (2021) study underscores the value of bringing KiP and Commognition into conversation. Their examples demonstrate how task design orients learners toward new discursive practices while simultaneously reorganizing intuitive resources. Commognition reveals the broader cultural-historical trajectories into which students are drawn, while KiP makes visible the micro-level dynamics of activation and reorganization in the moment. Considering the analysis through both perspectives clarifies how learners' reasoning is tuned within immediate activity and situated within longer-term discursive developments: KiP provides explanatory machinery for local dynamics of change, while Commognition situates these changes within the evolving architecture of mathematical discourse.

Example 2: From Configural to Deductive Discourse

The second study we discuss, Ben-Dor & Heyd-Metzuyanim (2021), illustrates how we may consider the Commognitive and KiP perspectives together, through an analysis of a group of middle-school students solving a geometry problem in a technologically enhanced environment. Their analysis reveals a coalescence between two different discourses, a configural discourse focused on visual features and a deductive discourse emphasizing formal reasoning. The study provides evidence of a moment in which a new shared discourse emerges through peer interaction. The data consists of detailed video analysis and transcript excerpts capturing discursive shifts and collaborative meaning-making.

Ben-Dor & Heyd-Metzuyanim reported that two students (Orna and Tamara) began with configural interpretations, relying on appearance. For instance, they justified their claims by saying things like *"they look equal"* or pointing to *"same-looking squares"* (Ben-Dor & Heyd-Metzuyanim, 2021, p. 127). In these moments, visual similarity was taken as sufficient evidence. As their discussion unfolded, however, the discourse shifted. Orna moved beyond judgments based on appearance by introducing algebraic notation, saying *"this is x (c) and this is x (d)"* (p. 128), while Tamara emphasized the importance of givens. Eventually, Orna reframed Tamara's configural procedure into a deductive one, insisting: *"you need to say that this is like (i.e., congruent to) that… in order for it to be ok to move…"* (Ben-Dor & Heyd-Metzuyanim, 2021, p. 128). This illustrates the transition from configural judgments of sameness to deductive justification grounded in congruence theorems. These moments capture a discursive change, a shared movement from judging by appearance to building a reasoned, mathematical argument. From a commognitive viewpoint, this is a moment where students blend two different ways of thinking: one based on how things look, and one based on formal properties. Their written statement reflects not just a conclusion, but a new kind of shared language, one that didn't exist when they began the task. But what happens inside this shift? What helps a student let go of what "looks right" and begin to trust what can be measured and reasoned?

A Knowledge in Pieces approach invites us to think about how this shift happens. Perhaps one student relies on an intuitive idea like "shapes that look the same probably are the same." This way of thinking might usually feel right. But then another idea, that one must rely on givens and measurable relations, becomes stronger through interaction and feedback. Could it be that these two ways of knowing were always there, but the moment of change comes when one becomes more important than the other? How do such shifts build over time and what makes them robust and enduring? Do they settle into a new way of seeing and reasoning, or do they remain unstable, depending on the context? An intuitive sense of congruency may stem from classroom experiences in which what is *seen* as equal in size is gradually connected to what is *reasoned* about through measurement and normative justifications. As a socially constructed experience, this provides an example of how intuitive knowledge can be grounded in classroom practice. Indeed, Sfard argues, “learning mathematics may be seen as transforming these spontaneously learned colloquial discourses rather than as building new ones from scratch” (Sfard, 2007, p. 573), which suggests that discourses that are “spontaneously learned” may have origins or grounding in physical experience.

NETWORKING KIP AND COMMOGNITION: A PROPOSAL

Knowledge in Pieces and Commognition differ in their focus on individual knowledge versus collective knowledge and its development, with Knowledge in Pieces theorizing more squarely on cognition of individuals in contexts and Commognition concerned with the interplay between individuals and their discursive communities in producing and reproducing knowledge over multiple timescales. While research informed by Knowledge in Pieces will continue to focus on the role of intuitive knowledge in reasoning processes and research informed by Commognition will continue to focus on the importance of others in negotiating and bridging incommensurate discourse, the example of Philip (2011) demonstrates the potential for taking a complex systems perspective on social and discursive phenomena. The examples of mathematical learning processes illustrate how we could widen the scope of interpretation of data by considering both perspectives together on the same data of in-the-moment reasoning processes.

Taking from KiP a general complex systems perspective would offer a powerful tool to describe processes of use and change for Commognition researchers. A Commognitive Complex Systems perspective (CCS) could then model discursive processes in terms of recognition and prioritization of elements of discourse, learned through individuals’ participation in their discursive communities and the feedback they get through these interactions. That is, much like physical intuitions that have a history of productive use get recognized and used more readily in reasoning about the physical world (what diSessa, 1993 describes in terms of structured priorities of systems), the same may be true of discursive elements and the way that social feedback can shape meaning and usage. Using the complex systems model and the notion of structured priorities could add specificity to how discourse change processes unfold from a Commognitive perspective. Ben-Dor & Heyd-Metzuyanim (2021) illustrate an example of commognitive bridging between discourses, emphasizing the possibility of continuity between discourses as opposed to abrupt change. Contextuality of knowledge/discourse use, non-linearity of learning processes, and continuity of change processes, with re-use of elements of previous knowledge/discourse, are shared foci of attention for both KiP and Commognition and could be modeled well by the complex systems of KiP.

On the other hand, KiP research can be stimulated by recognizing the contributions of conceptualizing mathematics as a discourse and mathematics learning as a discursive process. The interplay between knowledge as constructed by individuals and cultural-historical knowledge is not typically a focus of KiP research and considering socially-derived narratives as part of individuals’ knowledge systems, along with the theoretical machinery developed by commognitive researchers, can enrich the capacity to empirically account for mathematical reasoning processes.

If we imagine two funnels (see Figure 1): KiP looks through the narrow end, focusing sharply on the microgenetic reorganization of knowledge elements as learners encounter novelty; Commognition looks from the wide end, situating these reorganizations in the broader trajectory of mathematical discourse as a cultural entity. Seen together, the two funnels converge on a fuller account of reasoning and learning processes, one that honors both the cultural-discursive level of learning and the fine-grained individual processes through which new mathematical meanings are constructed. Theory from the KiP side may shed light on the linkages between the learners' in the moment reasoning and knowledge elements that have been grounded in physical or embodied experiences as well as those having origins that are social in nature. While from the Commognitive side theories may track how culturally shared meta-discourses link to in the moment reasoning.

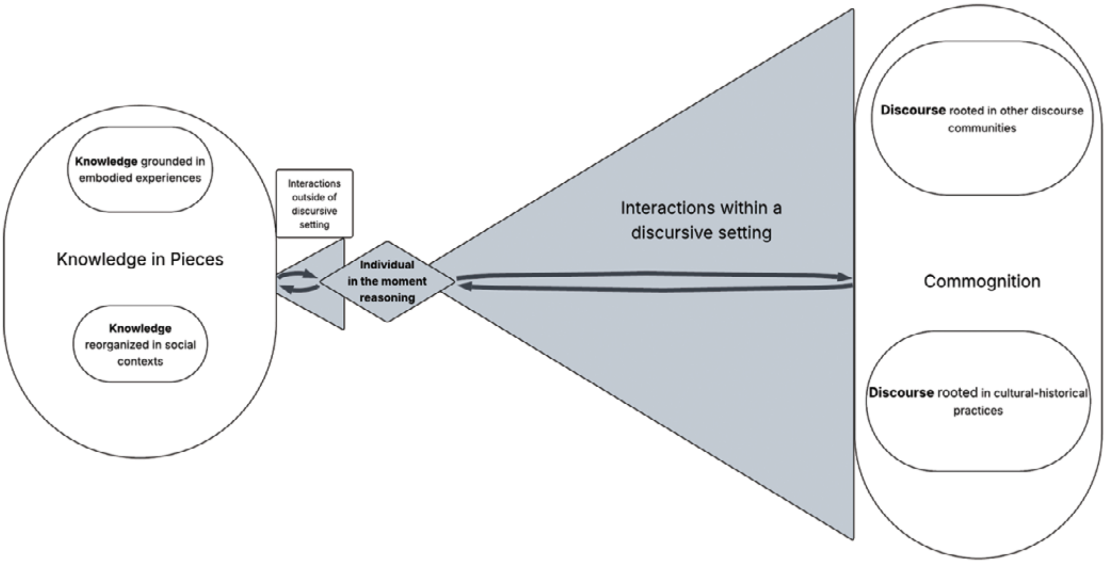

Figure 1. Relationship between KiP and Commognitive perspectives on in-the-moment reasoning

Figure 1 displays how both KiP and Commognition attend to in-the-moment reasoning. The long and short bidirectional arrows signify that the timescale over which the object of focus in the respective perspectives vary greatly, with Commognition emphasizing the long historical development of mathematics discourse rooted in the cultural historical practices of mathematics communities. Similarly, KiP has shorter arrows signifying that the objects of analysis are very local and particular to the individuals' nuanced ways of reasoning in a discursive setting. However, the individual lies slightly outside the discursive setting as well, bringing with them their embodied knowledge elements and other elements developed perhaps outside of physical experience. Finally, the size of the ovals containing the perspectives vary in size, signifying how the Commognitive perspective focuses on objects that are relevant to macro discourses (e.g., at the level of a mathematics community) while the KiP oval is smaller, signifying the focus being on the nuance of knowledge displayed by individuals regardless of physical or social origin.

As heuristic epistemological frameworks, both Knowledge in Pieces and Commognition are open to be adapted and extended to explain different aspects of thinking and learning than they were originally developed for. The joint perspective of CCS itself is flexible in terms of how it can be taken up. A commognitive researcher would not need to necessarily lay down the strong assumption about the social source of *some* mathematical knowledge or the refusal to model cognition in terms of mental constructs in order to follow a research direction that conceives of existing theoretical objects in commognition in terms of system dynamics. On the other hand, while KiP has traditionally focused on the nature and form of intuitive knowledge and its importance in both novice and expert reasoning processes, it does not claim that all knowledge has its source in physical experience. Thus, exploring knowledge in discursive terms can also be done through the lens of Knowledge in Pieces. The work of Philip (2011) and Kapon & diSessa (2012) introducing knowledge ontologies such as naturalized axioms and explanatory primitives, respectively, illustrates how Knowledge in Pieces can support such theorizing.

Given that analyzing data that reveals the nuances of moment-by-moment cognition are a strong commitment of both perspectives, this suggests one methodological principle for identifying when the use of both perspectives could be particularly fruitful would be to first identify moments of qualitative change in reasoning patterns in moment-by-moment data of learning interactions. In the moments identified, if a teacher or another individual appears as the impetus for shifting the local discourse, a KiP perspective could encourage attending to whether there are everyday or embodied experiences that may be connected with students' responses to these bids for discursive change.

DISCUSSION

In creating a dialogue between perspectives, one danger is oversimplification. When one values synthesis over all else, the issues that initially gave rise to the opposing positions can be erased and lose their impact. This is an issue to guard against in the proposed work of putting Knowledge and Pieces and Commognition in dialogue with each other. Knowledge in Pieces grew out of understanding the value of physical intuitions in both novice and expert reasoning processes, breaking down assumptions about a strong separation between the nature of novice knowledge and the nature of expert knowledge. It pushed researchers to recognize the substantial continuity between novices and experts as opposed to focusing on differences, treating novice knowledge from a deficit perspective. Similarly, Commognition grew out of Sfard's theorizing about how thinking patterns that emerge in the history of mathematics at a societal level appear to get regenerated at a personal level. Sfard's unifying approach of describing both social and personal mathematical knowledge in terms of a single type of construct, discourse, allowed for bringing together these very different time scales. More broadly, theoretical perspectives are intended to help us make sense of the multi-faceted and complex reality of what it means to know and how we learn. In this paper, we have proposed a way to think about using insights from KiP and Commognition, two perspectives that on the surface can seem to be in conflict with one another. Scheiner (2020) proposes four modes of dealing with opposing perspectives: (1) Claiming the perspectives are incommensurable, (2) Holding opposites not as conflicting but as complementary, (3) Dissolving or surpassing oppositions by blending perspectives or (4) Preserving paradoxes by recognizing the interdependence of constitutive oppositions. These modes build upon and extend the discussion of the relationship between theoretical positions described in diSessa, Levin & Brown (2016).

The incommensurability position holds that two perspectives are competing and incommensurable. There is no possibility of interaction between perspectives because they frame a common phenomenon in terms that are incompatible. Prior to this paper, it would be reasonable to argue that KiP and Commognition fell into this category. If instead, one can construe the perspectives as independent yet complementary, then there are a couple of different ways to proceed. For example, one could use both lenses on equal terms to look at the same phenomenon, getting a separate set of findings from each perspective. It could also be possible to operate on different levels of grain size and thus the perspectives could be construed as micro-complementary, to use a term from a categorization in diSessa, Levin and Brown. Complementary approaches reveal layers of understanding beyond what is possible with using a single approach. The third mode - dissolving or surpassing by blending perspectives - takes the oppositions between the perspectives and creates a blended perspective that has higher explanatory power. In the fourth mode, both points of contrast and connection are emphasized in building theory by moving back and forth between opposing perspectives. As opposed to dismissing or overcoming oppositions, this position embraces opposing perspectives in an ongoing interplay. The approach we have proposed to networking KiP and Commognition we argue is an example of approach four in Scheiner's taxonomy, *preserving paradoxes by recognizing the interdependence of constitutive oppositions*. At a very top level, it is possible to see the differences between Knowledge in Pieces and Commognition in terms of the source and nature of knowledge. A remaining tension can be articulated in the form of the question "Is the source of knowledge physical experience (diSessa/KiP) or is it social in nature (Sfard/Commognition)? To make progress on networking these theoretical perspectives, we relax the demand to settle the issue for *all* forms of knowledge. Our proposal is that for some learning phenomena it is possible to use the heuristics of Knowledge in Pieces, specifically the idea of modeling knowing and learning as a complex system of contextually-activated elements, to extend the theoretical machinery proposed by commognitive theorists. In this sketch, we propose one path forward, what we have termed the CCS, could involve starting from the basic objects of theorization in commognitive theory, discursive objects, and to consider them through the lens of a complex system. Modeling in this way, using the mechanism of shifting priorities of recognition and use of discursive elements through social feedback, can give theoretical tools to track gradual changes in discourse use over time. The analysis we discussed of personal ideologies and ideology change in the work of Philip (2011) illustrates how KiP can be used in this way.

We are arguing that this is genuinely new intellectual territory opened up by extending Commognition in this way and mathematics as discourse is a genuinely new kind of phenomenon for Knowledge in Pieces to tackle in the domain of mathematics. It is especially valuable to be able to connect Knowledge in Pieces with a perspective that accounts for the central problem tackled by Commognition, that is, how cultural knowledge is replicated and extended over time. There might continue to be questions about the extent to which knowledge has its roots in physical experience (diSessa) or roots in social experience (Sfard). However, the approach we propose here respects that the two traditions will continue to theorize according to their core assumptions, some of which are in tension with each other, while still enabling us to harness the power of each perspective for our larger goal of developing an understanding of the richness of mathematical cognition and learning.

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